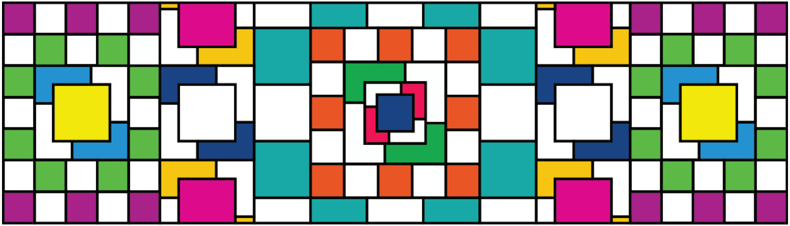
Multiscale Substitution Schemes and Kakutani Sequences of Partitions

Yotam Smilansky, Hebrew University of Jerusalem

Ergodic Theory and Dynamical Systems Seminar, University of Bristol



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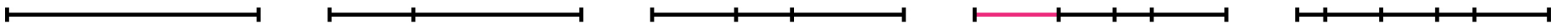

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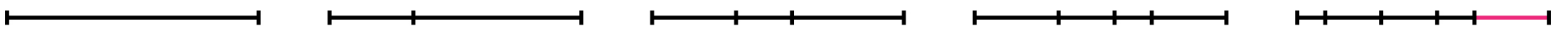

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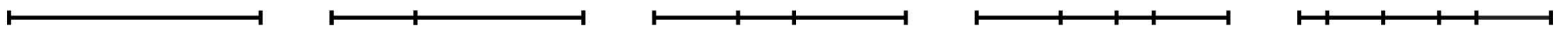

Kakutani's Theorem: For all $\alpha \in (0, 1)$, the α-Kakutani sequence of partitions is uniformly distributed in $\mathcal{I}$.

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In the $\frac{1}{3}$-Kakutani sequence, whenever a partition is made, color the shorter new interval red and the longer new interval blue:

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3. In case both limits exist, are they necessarily the same?

Multiscale substitution schemes

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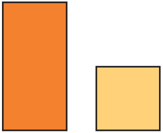

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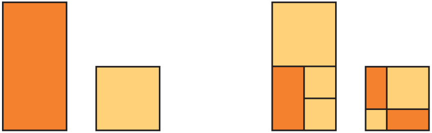

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2. **Substitution rule** assigning to every $\mathcal{T}_i$ a list of tiles

$$\mathcal{SR}\left(\mathcal{T}_i\right) = \left(\alpha_{ij}^{(k)}\mathcal{T}_j : j = 1, \ldots, n;\ k = 1, \ldots, k_{ij}\right)$$

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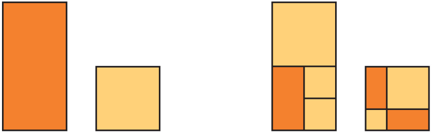

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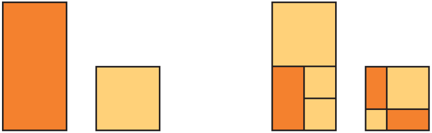

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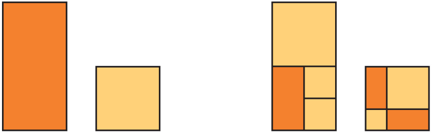

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Let $\mathcal{A}(\mathcal{T}_i)$ be the set of all labeled tiles which appear by applying the substitution finitely many times on $\mathcal{T}_i$ and subsequent tiles.

A scheme is **irreducible** if $\mathcal{A}(\mathcal{T}_i)$ contains tiles of type j for all i, j.

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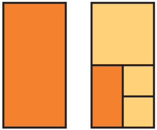

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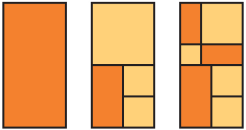

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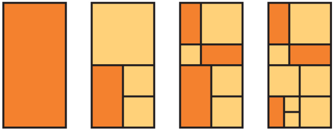

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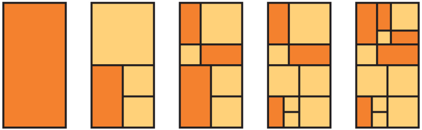

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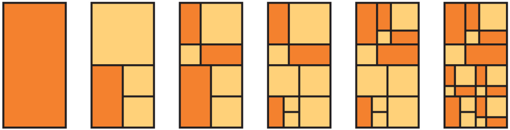

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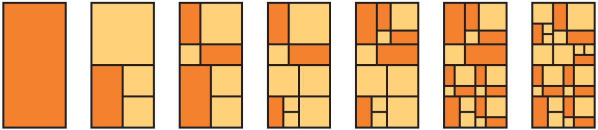

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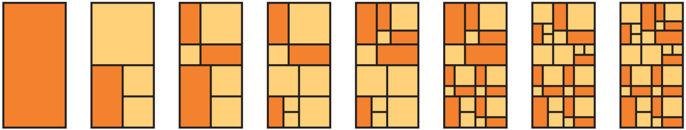

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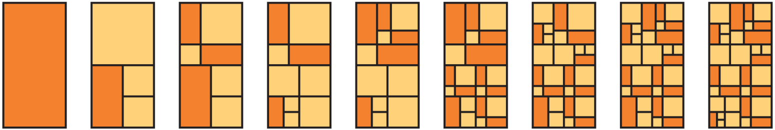

Example - The α-Kakutani sequence

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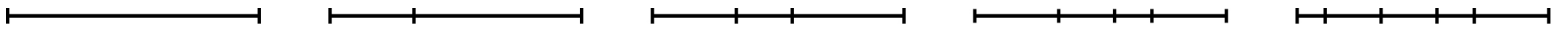

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This is equivalent to the weak-* convergence of the normalized sampling measures

$$\frac{1}{|x_n|} \sum_{x \in x_n} \delta_x$$

to the normalized Lebesgue measure on U, where δ_x is the Dirac measure concentrated at x.

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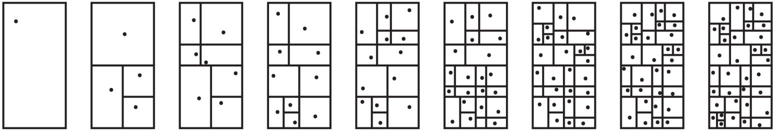

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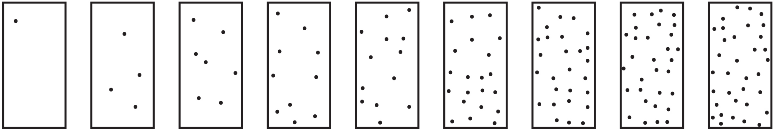

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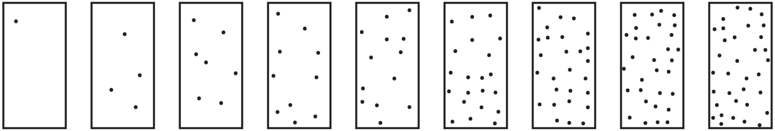

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Theorem

Let $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_n)$ be a set of prototiles and let $\{\pi_m\}$ be a Kakutani sequence of partitions of $\mathcal{T}_i \in \mathcal{F}$ generated by an irreducible multiscale substitution scheme on $\mathcal{F}$. Then $\{\pi_m\}$ is uniformly distributed in $\mathcal{T}_i$.

Tile counting argument implies uniform distribution

Lemma

Let $\{\gamma_m\}$ be a sequence of partitions of $\mathcal{T}_i \in \mathcal{F}$ generated by a multiscale substitution scheme on $\mathcal{F}$, such that for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ so all tiles in γ_m are of diameter less than ε for all $m \geq m_0$. Assume there exists a marking sequence $\{x_m\}$ of $\{\gamma_m\}$ such that for any tile $T \in \mathcal{A}(\mathcal{T}_i)$

$$\lim_{m\to\infty} \frac{|\{x_m \cap T\}|}{|x_m|} = \frac{\mathrm{vol}\, T}{\mathrm{vol} \mathcal{T}_i}.$$

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- Counting of tiles is done using **directed weighted metric graphs**.

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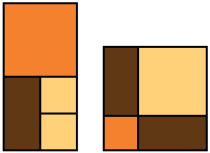
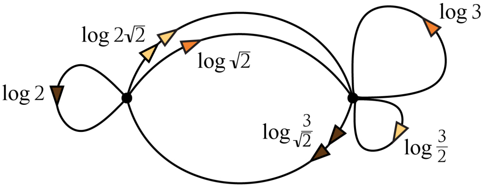
   - Weight of ε is $l(\varepsilon) = \log \frac{1}{\alpha}$.

Graphs associated with multiscale substitution schemes

The directed weighted metric graph $G = (\mathcal{V}, \mathcal{E}, l)$ associated with a multiscale substitution scheme on $\mathcal{F} = \{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$ has

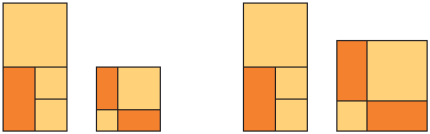
1. $\mathcal{V} = \{1, \ldots, n\}$, vertex $i \in \mathcal{V}$ is associated with prototile $\mathcal{T}_i$.
2. For every $\alpha\mathcal{T}_j \in \mathcal{SR}(\mathcal{T}_i)$ there is an edge $\varepsilon \in \mathcal{E}$ such that
   - Initial vertex of ε is $i \in \mathcal{V}$.
   - Terminal vertex of ε is $j \in \mathcal{V}$.
   - Weight of ε is $l(\varepsilon) = \log \frac{1}{\alpha}$.





Equivalent and normalized schemes

Two schemes on $\mathcal{F}^1 = (\mathcal{T}_1, \ldots, \mathcal{T}_n)$ and $\mathcal{F}^2 = (\lambda_1\mathcal{T}_1, \ldots, \lambda_n\mathcal{T}_n)$ are **equivalent** if the substitution rules are the same up to rescaling.



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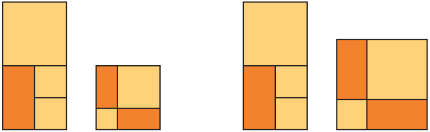

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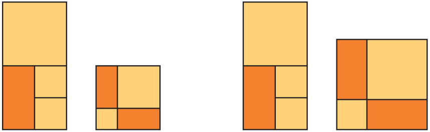

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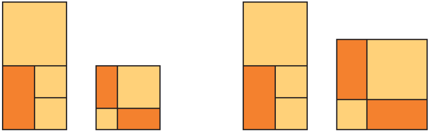

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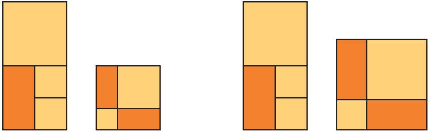

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If the scaling constants in a normalized scheme are β_{ij}, then for every equivalent scheme the scaling constants are

$$\alpha_{ij} = \left(\frac{\mathrm{vol} \mathcal{T}_i}{\mathrm{vol} \mathcal{T}_j} \right)^{1/d} \beta_{ij}.$$

The β_{ij}'s are called the **constants of substitution**.

Paths in G and tiles in $\mathcal{A}(\mathcal{T}_i)$

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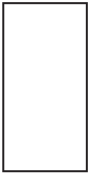

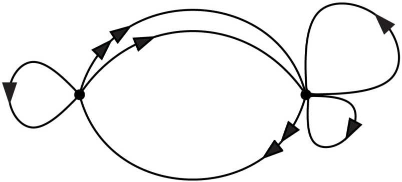

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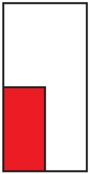

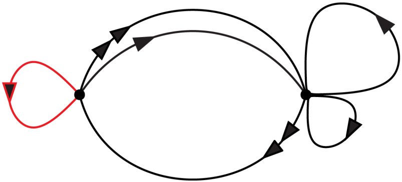

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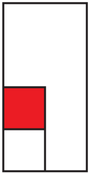

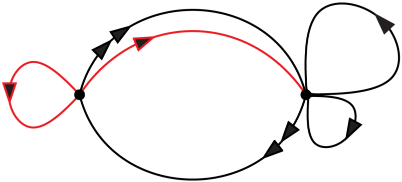

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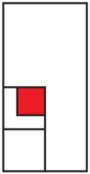

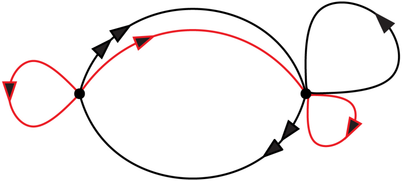

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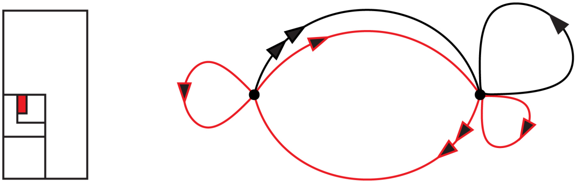

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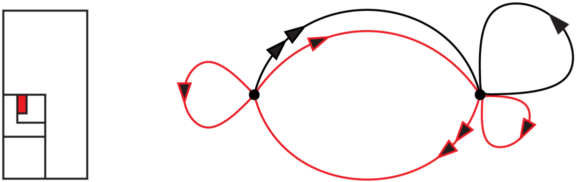

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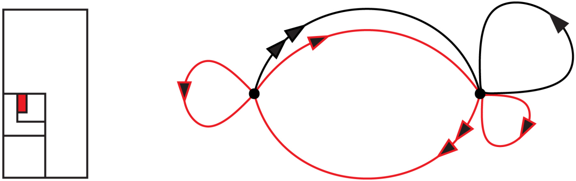

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- Tiles of maximal volume in π_m are associated with paths of length l_m, where $\{l_m\}$ is the increasing sequence of lengths of paths in G with initial vertex $i \in \mathcal{V}$.

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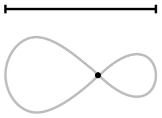

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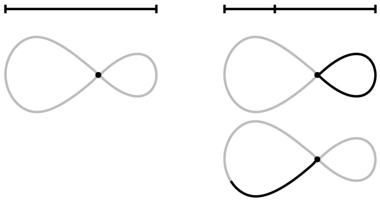

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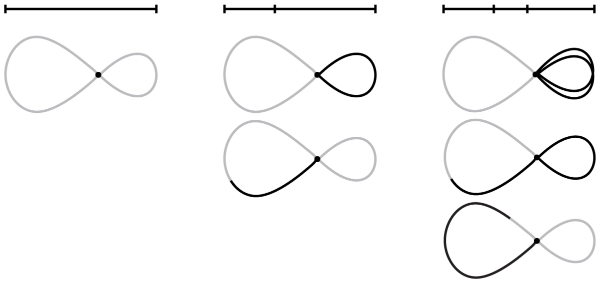

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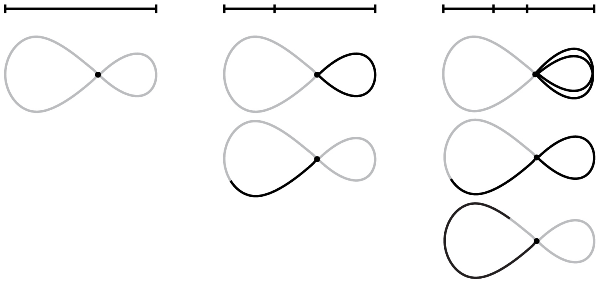

- Counting tiles in π_m is reduced to counting metric paths of length l_m in the associated graph.

Incommensurable and commensurable schemes

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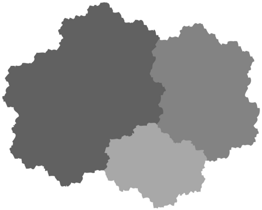
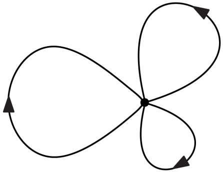
A commensurable example – **The Rauzy fractal scheme:**

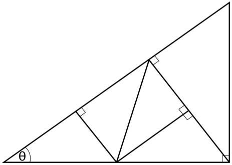


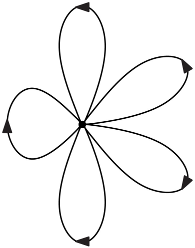


Edge lengths: $\log \tau, 2 \log \tau, 3 \log \tau$, where $\tau =$ tribonacci constant.

Sadun's generalized pinwheel

For a.e θ the generalized pinwheel scheme is incommensurable:





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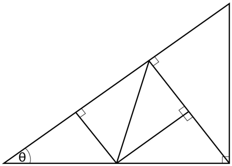

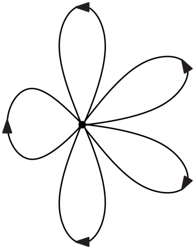

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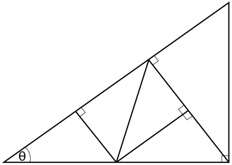

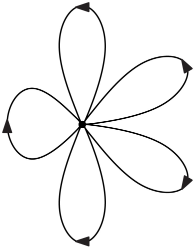

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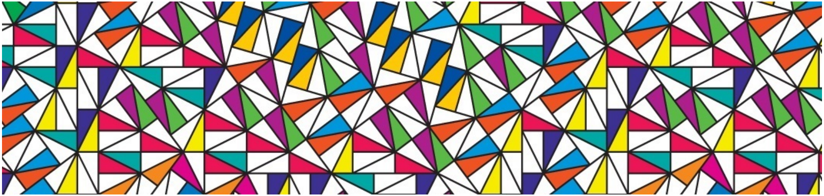

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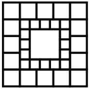
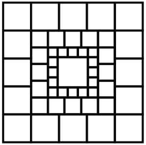
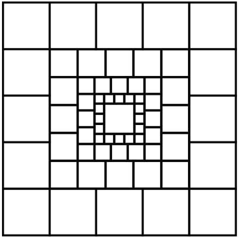
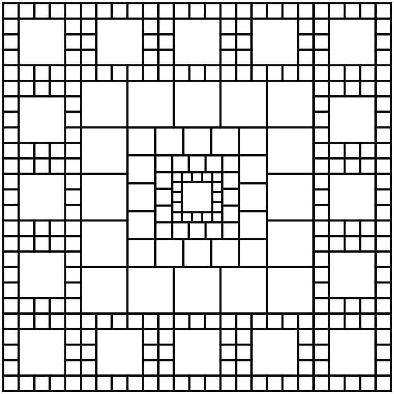
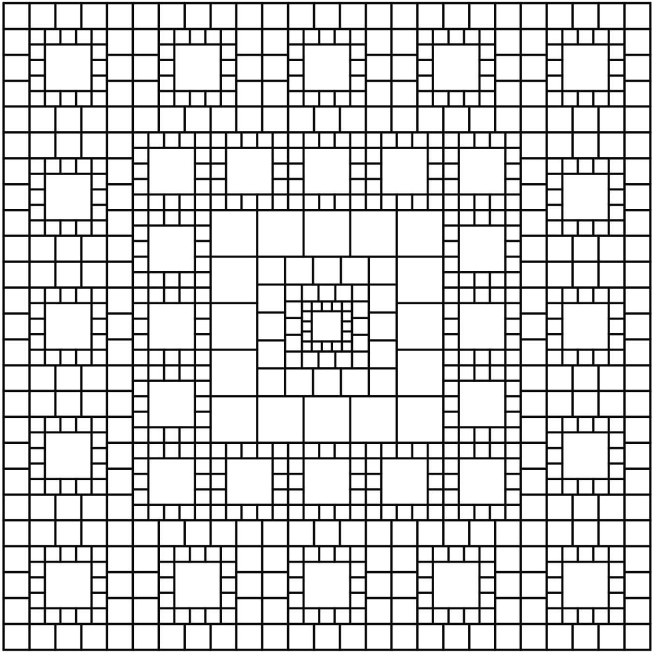
The **tiling space** X_H is the space of all tilings τ of $\mathbb{R}^d$ with the property that every patch of τ is a limit of translated sub-patches of elements of $\mathscr{P} = \cup \mathscr{P}_i$. Elements of X_H are called **multiscale substitution tilings**.

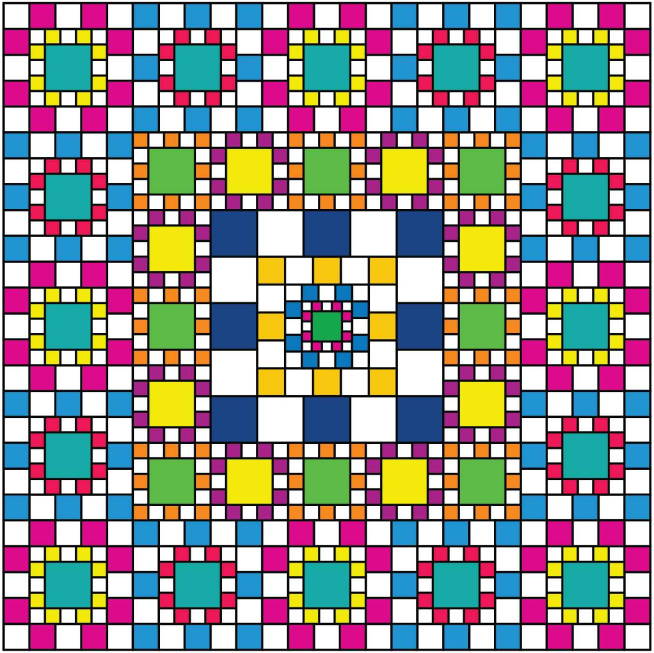












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- Many more beautiful properties! *Coming soon...*

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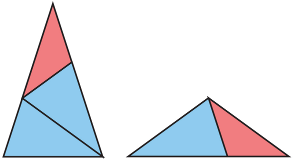
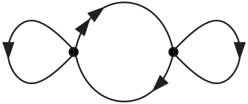
Example: The Penrose-Robinson substitution $\alpha = \frac{1}{\varphi}$:

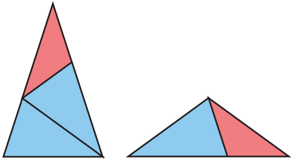


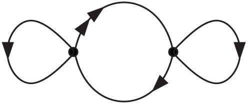


The commensurable case - fixed scale schemes

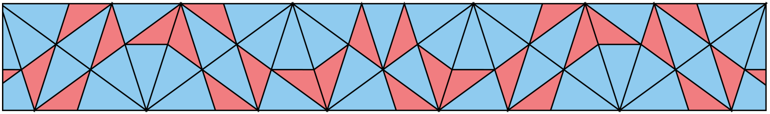
If $\alpha_{ij} = \alpha \in (0,1)$ for all i and j the scheme is **fixed scale**.

Example: The Penrose-Robinson substitution $\alpha = \frac{1}{\varphi}$:





This is the classical setup for **substitution tilings** of $\mathbb{R}^d$:



The commensurable case - generations sequences

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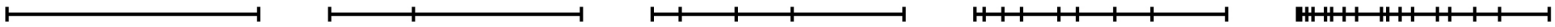

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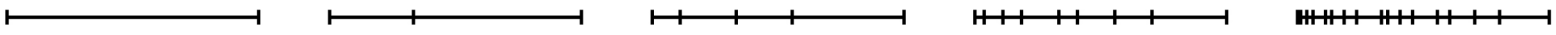

Theorem

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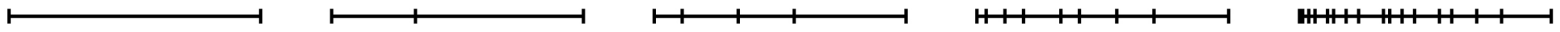

Theorem

Generations sequences of partitions generated by fixed scale substitution schemes are uniformly distributed.

Follows from the Perron-Frobenius Theorem for irreducible matrices, and additional standard results on cyclic matrices.

The commensurable case - Kakutani vs. generations

Lemma

*Any **Kakutani** sequence of partitions generated by a commensurable scheme is a subsequence of a **generations** sequence of partitions generated by some fixed scale scheme.*

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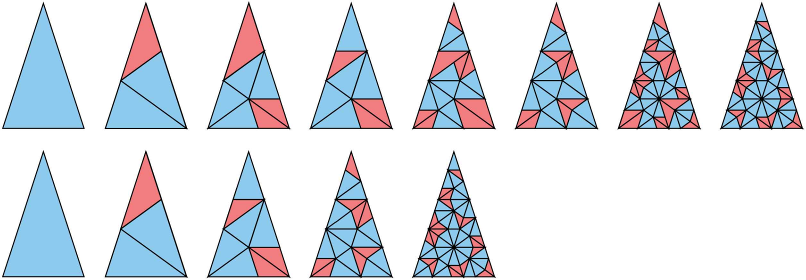

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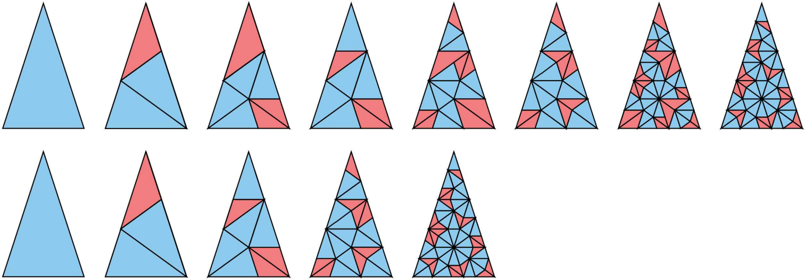

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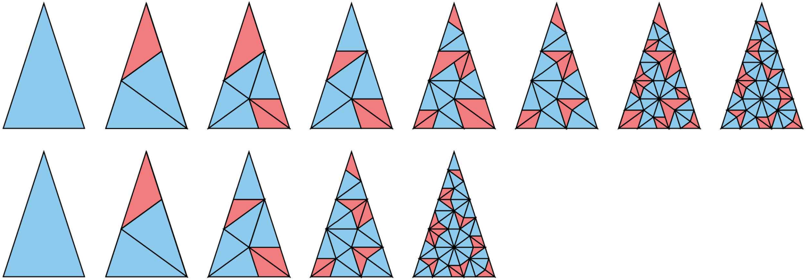

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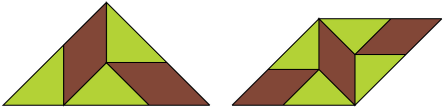
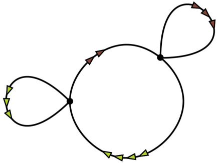
Sketch of proof of lemma – "slowing down"





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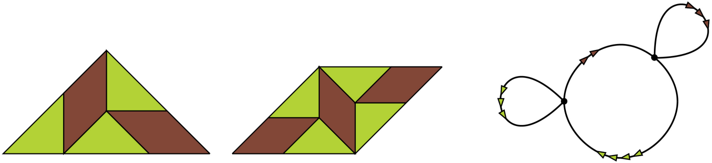

Claim: There exist equivalent schemes such that:

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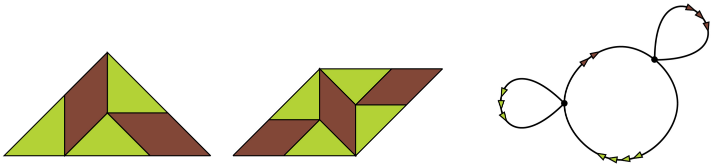

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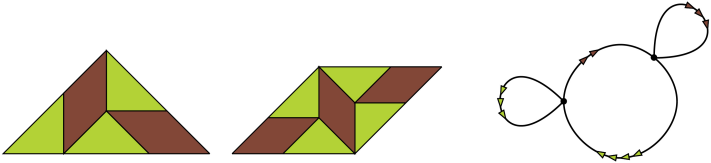

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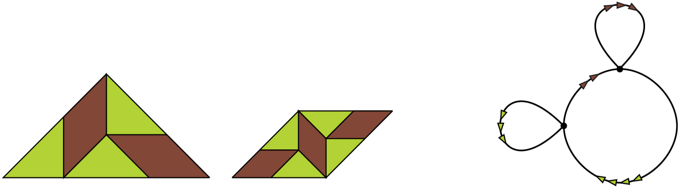
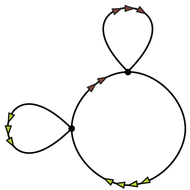
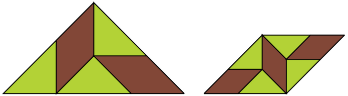


Defining a fixed scale scheme

We can now add vertices to G in a way that all edges in the new graph are of **equal length**





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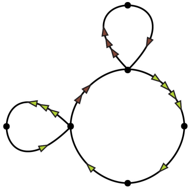
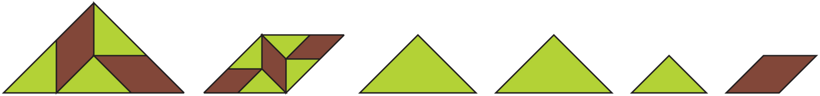


This graph is associated with a fixed scale scheme on the prototiles:



The graph matrix function

Let $\varepsilon_1, \ldots, \varepsilon_{k_{ij}}$ the edges with initial vertex i and terminal vertex j.

The **graph matrix function** of G is the matrix valued function $M : \mathbb{C} \to M_n(\mathbb{C})$ defined by

$$M_{ij}(s) = e^{-s \cdot l(\varepsilon_1)} + \cdots + e^{-s \cdot l\left(\varepsilon_{k_{ij}}\right)},$$

and $M_{ij}(s) = 0$ if there are no such edges in G.

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Lemma

Let G be a graph associated with an irreducible multiscale substitution scheme in $\mathbb{R}^d$. Then $M(d)$ is a non-negative irreducible matrix with a positive Perron-Frobenius right eigenvector

$$v_{PF} = v_{vol} = (\mathrm{vol}\mathcal{T}_1, \ldots, \mathrm{vol}\mathcal{T}_n) \in \mathbb{R}^n$$

and Perron-Frobenius eigenvalue $\mu_{PF} = 1$.

Counting paths on incommensurable graphs

Theorem ([Kiro, Smilansky×2 (2018)])

Let G be a strongly connected incommensurable graph. There exist $\lambda > 0$ and $Q \in M_n(\mathbb{R})$ with positive entries, such that if $\varepsilon \in \mathcal{E}$ has initial vertex $h \in \mathcal{V}$, the number of metric paths of length exactly x from vertex $i \in \mathcal{V}$ to a point on the edge ε grows as

$$\frac{1 - e^{-l(\varepsilon)\lambda}}{\lambda} Q_{ih} e^{\lambda x} + o\left(e^{\lambda x}\right), \quad x \to \infty.$$

where λ is the maximal real value for which $\rho\left(M\left(\lambda\right)\right) = 1$,

$$Q = \frac{\mathrm{adj}\left(I - M\left(\lambda\right)\right)}{-\mathrm{tr}\left(\mathrm{adj}\left(I - M\left(\lambda\right)\right) \cdot M'\left(\lambda\right)\right)}.$$

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It follows that $M(\lambda)$ is a non-negative irreducible matrix with Perron Frobenius eigenvalue $\mu_{PF} = 1$, and the columns of Q are spanned by an associated Perron Frobenius eigenvector v_{PF}.

Tile counting and uniform distribution

Theorem

Let $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_n)$ be a set of prototiles in $\mathbb{R}^d$ and let $\{\pi_m\}$ be a sequence of partitions of a tile $\mathcal{T}_i$ generated by an irreducible incommensurable multiscale substitution on $\mathcal{F}$. Then

$$|\{\mathrm{Tiles} \in \pi_m\}| = \sum_{j=1}^{n} \sum_{h=1}^{n} \sum_{k=1}^{k_{hj}} \frac{1 - \left(\beta_{hj}^{(k)}\right)^d}{d} q_h e^{dl_m} + o\left(e^{dl_m}\right), \quad m \to \infty,$$

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Proof of uniform distribution $\{\pi_m\}$: Let $T \in \mathcal{A}(\mathcal{T}_i)$, say T appears at partition π_{m_0} and is of type r.

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$$\frac{|\{x_m \cap T\}|}{|x_m|} = \frac{|\{\mathrm{Tiles} \in \widetilde{\pi}_{m-m_0}\}|}{|\{\mathrm{Tiles} \in \pi_m\}|} = \frac{e^{d\left(l_m - l_{m_0}\right)}}{e^{dl_m}} + o(1),$$

and since $e^{-l_{m_0} d} = \frac{\mathrm{vol}\, T}{\mathrm{vol} \mathcal{T}_i}$ uniform distribution follows.

Elements of the proof of the path counting formula

The Wiener-Ikehara Theorem: Let $f(x)$ be a non-negative and monotone function on $[0,\infty)$, such that

$$\mathcal{L}\left\{f\left(x\right)\right\}\left(s\right)=\int_0^{\infty} f\left(x\right)e^{-xs}dx,$$

converges for $\mathrm{Re}(s)>\lambda$, and $\mathcal{L}\left\{f\left(x\right)\right\}\left(s\right)-\frac{c}{s-\lambda}$ extends to a continuous function in $\mathrm{Re}(s)\geq\lambda$, then

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and our theorem follows from a study of the locations of the zeroes of the exponential polynomial $\det\left(I-M\left(s\right)\right)$.

A simple but interesting example

Graphs associated with incommensurable α-Kakutani or pinwheel schemes have a single vertex and loops of two lengths. We get

$$\mathcal{L}\left\{B_{i,\varepsilon}(x)\right\}(s) = \frac{1}{1 - e^{-as} - e^{-bs}}$$

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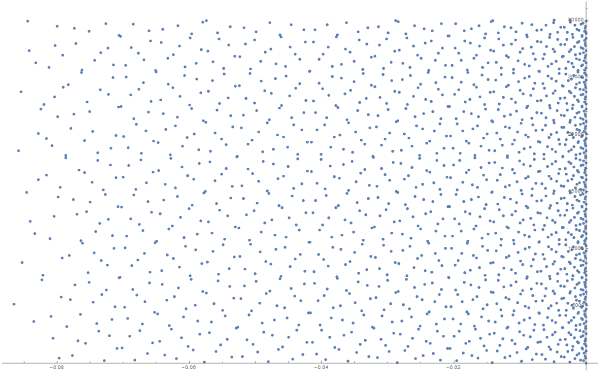
For example if $\frac{a}{b} = \sqrt{2}$, numerics show that the pole structure is



The incommensurable case - types and their frequencies

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Under the previous assumptions

$$\frac{|\{\text{Tiles of type } r \text{ in } \pi_m\}|}{|\{\text{Tiles in } \pi_m\}|} = \frac{\sum_{h=1}^{n} q_h \sum_{k=1}^{k_{hr}} \left(1 - \left(\beta_{hr}^{(k)}\right)^d\right)}{\sum_{r=1}^{n} \sum_{h=1}^{n} q_h \sum_{k=1}^{k_{hr}} \left(1 - \left(\beta_{hr}^{(k)}\right)^d\right)} + o(1).$$

The incommensurable case - asymptotic volumes

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2. Special properties of graphs and the relevant probabilities which are associated with substitution schemes.

A nice answer

Back to the red and blue $\frac{1}{3}$-Kakutani sequence:

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2. $\lim\limits_{m\to\infty} \operatorname{vol}\left(\bigcup \{\text{Red intervals } \in \pi_m\}\right)$

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2. $\lim_{m\to\infty} \mathrm{vol}\left(\bigcup \{\text{Red intervals } \in \pi_m\}\right) = \frac{\frac{1}{3}\log\frac{1}{3}}{\frac{1}{3}\log\frac{1}{3} + \frac{2}{3}\log\frac{2}{3}}$.

Thanks!